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LETTER TO THE EDITOR

Distillable entanglement in $d \otimes d$ dimensions

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Abstract
Distillable entanglement (E_d) is one of the acceptable measures of entanglement of mixed states. On the basis of discrimination through local operation and classical communication, this letter gives E_d for two classes of orthogonal multipartite maximally entangled states.

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1. Introduction

Entanglement is believed to be a genuine resource for quantum computation and teleportation. Entanglement shows up in composite quantum systems where subsystems do not have *pure* states of their own. This is a strict quantum phenomenon with no classical analogue. However, it is highly nontrivial to keep such entangled states away from the effect of the environment. The environment destroys 'part' of such entanglement. The residual entanglement could be purified or distilled to give maximally entangled states. The maximally entangled states (MES), among which there are singlet states, can be taken as the basic unit to quantifying entanglement. The distillation process uses only local operation (e.g., unitary transformation) and classical communication (e.g., phone calls) known as LOCC. Different LOCC processes could give different number of singlet MES or states that could be transformed into singlet state. The maximum number of singlet MES is the *distillable entanglement*. Distillable entanglement is of major importance in quantum information processing; the transmission of a pure state through a noisy channel introduces noise [1] to the initial pure state and the state becomes mixed. Therefore, it is important to know the distillable entanglement of the state after transmission.

For pure bipartite states there is a widely acceptable measure of entanglement, the Von Neumann entropy. Whereas for mixed states[3] different quantities are proposed as a measure of the degree of entanglement. Few of these quantities survive nowadays, among the acceptable degrees of entanglement is the distillable entanglement E_d and the relative entropy

[3] A mixed state is entangled if it cannot be represented as a mixture of unentangled pure states.

of entanglement E_r [2]. The latter is an upper bound for the former [3, 4]. This upper bound will be used to calculate E_d. Hence our strategy in evaluating the distillable entanglement will be

$$\text{Number of singlet MES } \leqslant E_d \leqslant E_r.$$

To saturate the lower bound we use discrimination of *orthogonal* MES by LOCC as a distillation procedure. It should be mentioned that any orthogonal states $|\psi\rangle$ and $|\phi\rangle$ could be discriminated perfectly by *global* measurement, since they satisfy the necessary condition for discrimination $\langle\phi|\psi\rangle = 0$. Walgate *et al* [5] have shown that two orthogonal *pure* states can always be discriminated by LOCC without any need for global measurements. However, two orthogonal *entangled* states can be discriminated by LOCC if only one copy is provided. Recently, Ghosh *et al* [6] have generalized this result by using the teleportation protocol of [7], and proved that d pairwise orthogonal MES in $d \otimes d$ (spin $(d-1)/2$ particles) can always be discriminated by LOCC with a single copy provided. Following the outlined strategy of saturating the lower and upper bounds for E_d and on the basis of the above results of discrimination in $d \otimes d$, we will consider two classes of mixtures of orthogonal MES in $d \otimes d$, and evaluate their E_d.

2. Four-party state

The first class of states we consider is the generalization of the state studied in [8]:

$$\rho = \frac{1}{d}\sum_{i=1}^{d} |\Psi_i\rangle\langle\Psi_i|_{AB} \otimes |\Psi_i\rangle\langle\Psi_i|_{CD} \tag{1}$$

where the states $|\Psi_i\rangle$ are any d pairwise orthogonal MES chosen out of the d^2 orthogonal MES, $\{|\psi_{nm}\rangle; n, m = 0, 1, \ldots, d-1\}$, in $d \otimes d$ defined by [7]

$$|\psi_{nm}\rangle = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1} \mathrm{e}^{2\pi \mathrm{i} jn/d}|j\rangle \otimes |(j+m)\mathrm{mod}\, d\rangle \tag{2}$$

where $\{|j\rangle; j = 0, 1, \ldots, d-1\}$ is the standard orthogonal basis of the d-dimensional Hilbert space. ρ could be understood as a four-party state where A and B share one of the d orthonormal MES, but do not know which one (all terms are equally weighted by $1/d$), and C and D share the same MES, also not knowing which state they are sharing.

As in [9], in order to compute the distillable entanglement of ρ we will use the relative entropy of entanglement for an entangled quantum state σ defined as [2]

$$E_r(\sigma) = \min_{\sigma^* \in \mathcal{D}} S(\sigma||\sigma^*) \tag{3}$$

where $\mathcal{D}$ is the set of all separable states on the Hilbert space on which σ is defined and $S(\sigma||\sigma^*) = \mathrm{Tr}\{\sigma(\log_2\sigma - \log_2\sigma^*)\}$ is the relative entropy of σ with respect to σ^*.

The relative entropy of entanglement for ρ could be found by computing the relative entropy of ρ with respect to the equal combination of all the d^2 MES:

$$\rho^S = \frac{1}{d^2}\sum_{i=1}^{d^2} |\Psi_i\rangle\langle\Psi_i|_{AB} \otimes |\Psi_i\rangle\langle\Psi_i|_{CD}. \tag{4}$$

We will show that ρ^S minimizes $S(\rho||\rho^*)$ over $\rho^* \in \mathcal{D}$. What is particular about ρ^S is that it is separable across all two-party cuts, $AC : BD$, $AD : BC$ and by construction across $AB : CD$ (see [8] for the bipartite case, the generalization to multipartite case is straightforward). Hence the relative entropy of entanglement for ρ across $AC : BD$ cut is

$$E_r(\rho) \leqslant S(\rho||\rho^S) = \log_2 d. \tag{5}$$

However the distillable entanglement is bounded above by $E_r(\rho)$ [3, 4], so $E_d(\rho) \leqslant \log_2 d$. But it has been shown [6] that it is possible to distinguish between the d orthogonal MES by only LOCC if one copy of the state is provided. C and D can then know which state they have by LOCC and hence enable A and B to know with certainty which Bell state they share, which could then be transformed by local operation into a singlet state; this corresponds to $\log_2 d$ ebit[4], hence the distillable entanglement of ρ in $AC : BD$ cut is at least $\log_2 d$ ($E_d(\rho) \geqslant \log_2 d$). From the lower and upper bounds for E_d we deduce that

$$E_d(\rho) = \log_2 d. \tag{6}$$

This result is a generalization for the bipartite case

$$\rho = \frac{1}{2}\sum_{i=1}^{2} |\Psi_i\rangle\langle\Psi_i|_{AB} \otimes |\Psi_i\rangle\langle\Psi_i|_{CD} \tag{7}$$

where the state $|\Psi_i\rangle$ is one of the four Bell states

$$\begin{aligned} \psi_{00} &= \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad & \psi_{01} &= \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\ \psi_{11} &= \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \qquad & \psi_{10} &= \tfrac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \end{aligned} \tag{8}$$

and the distillable entanglement in this case is $E_d(\rho) = 1$ ebit as found in [9].

3. Multi-copy Bill state

The second class of state we will study is the multi-copy Bill state defined by

$$\rho_n = \frac{1}{d^2}\sum_{i=1}^{d^2} |\Psi_i\rangle^{\otimes n}\langle\Psi_i| \tag{9}$$

where each biparty among the n-biparties[5] shares one of the d^2 orthogonal MES, with an equal probability. To compute the distillable entanglement of ρ_n we will use the method described in [10] (for the $2 \otimes 2$ case).

For $n = 1, 2$ the state ρ_n is separable [8] and hence the distillable entanglement is zero. When n is even, $n = 2m$ with $m > 1$, the separable state ${\rho_2}^{\otimes m}$ could be used to optimize the relative entropy of ρ_{2m}. Indeed,

$$E_d(\rho_{2m}) \leqslant S(\rho_{2m}||{\rho_2}^{\otimes m}) = (2m-2)\log_2 d. \tag{10}$$

However, it has been shown [6] that two copies are necessary and sufficient to discriminate between d^2 orthogonal MES and hence any two biparties from the $2m$ biparties can use their two copies of $|\Psi_i\rangle$ to determine which state they have initially shared and hence distill $(2m-2)\log_2 d$ ebit between the remaining $2m-2$ biparties. So the distillation entanglement is at least $(2m-2)\log_2 d$. Combining this with equation (10) we get

$$E_d(\rho_{2m}) = (2m-2)\log_2 d. \tag{11}$$

For odd n, $n = 2m + 1$, an additional step is needed. We first use the fact that one can always decompose $\rho_{2m+1}^{\otimes 2}$ into two copies of the optimal decomposition of ρ_{2m+1}, i.e.

[4] For the bipartite case a 1 ebit describes any quantum system which contains entanglement equivalent to that of a singlet. However, a multipartite MES of two subsystems A and B has d equally weighted terms in its Schmidt decomposition, giving $\log_2 d$ of entanglement.

[5] not to be confused with bipartite.

$2E_r(\rho_{2m+1}) \leqslant E_r\big(\rho_{2m+1}^{\otimes 2}\big)$. Then the relative entropy of $\rho_{2m+1}^{\otimes 2}$ is evaluated with respect to ${\rho_2}^{\otimes 2m+1}$, which is separable:

$$E_r\big(\rho_{2m+1}^{\otimes 2}\big) \leqslant S\big(\rho_{2m+1}^{\otimes 2}||{\rho_2}^{\otimes 2m+1}\big) = (4m-2)\log_2 d. \tag{12}$$

Hence, the distillable entanglement of ρ_{2m+1} is bounded above as

$$E_d(\rho_{2m+1}) \leqslant E_r(\rho_{2m+1}) \leqslant \tfrac{1}{2}S\big(\rho_{2m+1}^{\otimes 2}||{\rho_2}^{\otimes 2m+1}\big) = (2m-1)\log_2 d. \tag{13}$$

Again, using two copies, out of the $2m+1$ copies, to distinguish between the $d^2|\Psi_i\rangle$ we get the lower bound

$$E_d(\rho_{2m+1}) \geqslant (2m+1-2)\log_2 d = (2m-1)\log_2 d.$$

The lower and upper bounds for E_d give

$$E_d(\rho_{2m+1}) = (2m-1)\log_2 d. \tag{14}$$

Here we do not venture to conjecture any statement concerning the additivity of the relative entropy of entanglement for the multipartite case, which will be investigated further in a future work.

Finally, combining the even and odd n cases, the distillable entanglement of ρ_n in $d \otimes d$ is

$$E_d(\rho_n) = (n-2)\log_2 d. \tag{15}$$

This is a generalization of the $d = 2$ case, studied in [10], where $E_{d=2}(\rho_n) = n-2$. Again discrimination is shown to be an optimal distillation procedure.

4. Conclusion

In this letter we have made a generalization to $d \otimes d$ of the results found in [9, 10] for the distillable entanglement of the four-party state equation (1) and the multi-copy Bell state equation (9), respectively. Discrimination was used as distillation procedure and shown to be optimal for the specific studied classes of states. A natural extension to our work would be the study of more than d states for the first class of states we used. For more than d state mixtures discrimination is no longer a good candidate for distillation (a mixture of $d+1$ states cannot be discriminated with one copy provided [6]). Another important issue is the distillation entanglement of orthogonal partially entangled mixtures, which will be addressed in a future work.

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